Solutions: *Challenge Set 3*

1. Plugging-in, we get (2/3)/(3/2). Dividing 2/3 by 3/2, we multiply by the reciprocal to get 2/3 x 2/3 = **4/9**.

2. It is slightly easier to calculate the white space and subtract it from the total area (9). The area of the smaller white triangle is (1 x 1)/2 = ½ and the larger triangle is (2 x 2)/2 = 2. 9 – 2 – ½ = **6½**.

3. A common method is to pick a number like 11 and square it, then divide by 7 and take the remainder (2). To understand why this works for all numbers requires a bit of knowledge of modular arithmetic. If a number x leaves a remainder of 4 when divided by 7, we say that it is congruent to 4 modulo 7: $x \equiv 4 (\text{mod} 7)$. For all $a \equiv b (\text{mod } n)$, $a^2 \equiv b^2 (\text{mod } n)$ … so $x^2 \equiv 4^2 (\text{mod } 7)$, but a number cannot leave a remainder of $4^2 = 16$ when divided by 7, so we divide out the remaining 7's to get a remainder of **2**.

4. The probability that he will make the first then miss the second is the same as the probability he will miss then make … so we calculate this probability and double it. P(make then miss) = (3/5)(2/5) = 6/25. Double this to get 12/25. We can also calculate the probability that he will make both (9/25) or miss both (4/25) and subtract these from 1, which also gives us the correct answer **12/25**.

5. There are $_4C_2$ = 6 ways to arrange two 1's among two 2's. If we add these in the usual way, the sum of each column will include three 1's and three 2's. 1 + 1 + 1 + 2 + 2 + 2 = 9, so each column will have a sum of 9 and we get 9,999. Alternatively, just write all six numbers and add them: 1,122 + 1,212 + 1,221 + 2,112 + 2,121 + 2,211 = **9,999**.

6. Instead of imagining the two runners approaching each other, one at 2m/s and the other at 3m/s, we begin by imagining one stationary and the other running 5m/s. When they are first 50m apart, the 50m is the hypotenuse of a 30-40-50 right triangle, so the two are 'horizontally' 40 meters apart and 'vertically' 30m apart. After the runner we now imagine running 5m/s (let's say Thomas) runs 40 meters, he will be directly across from Kyra. After 40 more meters, we can again see that they are 40m 'horizontally' and 30m 'vertically', forming another 30-40-50 triangle (at which point they are 50m apart again). Altogether Thomas ran 80m at 5m/s which takes 80/5 = **16** seconds.

7. There is a shortcut for adding the first n odds that we will use … the sum of the first n odd integers is equal to n^2 (this is easy to prove*). For example, the sum of the first four odds 1 + 3 + 5 + 7 = 16 = 4^2 (though not necessary in this problem, you are more likely to make an addition error adding nine odds compared to calculating 9^2 in your head.) The greatest value we can achieve is the sum of the first 9 odds, which is 81. The largest prime less than 81 is 79, but we cannot get 79 (there is no 2 to remove). We can get the next largest prime, **73**, by removing a 3 and a 5 (or a 1 and a 7) from the set and adding the rest.

8. The greatest perimeter is a long 3x70 rectangle made up of all ten rectangles laid end-to-end (perimeter = 146). The smallest perimeter is created by making the figure as close to a square as possible. In this case we have two rows of five horizontal rectangles to make a 14x15 rectangle of perimeter 58. 146-58=**88**.

9. Another useful formula: the sum of cubes $1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = (1 + 2 + 3 + 4 + \ldots + n)^2$, (this is tricky to prove**) so we have:

$$2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 = (1 + 2 + 3 + 4 + \ldots + 8)^2 - 1 = \left(\frac{8(9)}{2}\right)^2 - 1 = 36^2 - 1^2 =$$

(36 + 1)(36 – 1) = (37)(35), so **37** is the largest prime factor.

10. The number of the diagonals is n(n-3)/2 (look up why) and the sum of the interior angles is 180(n-2), so we write the equation n(n-3)/2 = 180(n-2) – 1. Multiply the whole thing by 2 to remove the denominator on the left to get n(n-3) = 360(n-2) – 2.

Now … two ways …

$n^2 - 3n = 360n - 722$, so

$n^2 - 363n + 722 = 0$, which factors into

(n-361)(n-2) = 0, and since a polygon cannot have 2 sides, 361 is the only solution.

Or … (suggested by students: first by Peter Luo and then by Roy Li and Pranay Orugunta) …

$n^2 - 3n = 360(n-2) - 2$, so

$n^2 - 3n + 2 = 360(n-2)$, which factors on the left into

(n-2)(n-1) = 360(n-2), and we divide both sides by (n-2), leaving:

n-1 = 360, so n = **361**.

:)

* The nth positive odd integer can be expressed 2n-1 (for example, the 5^{th} odd integer is 2(5)-1 = 9.) If we are adding 1 + 3 + 5 + … + 2n-1, the average of the n terms is the same as the average of the first and the last terms: [1 + (2n-1)]/2 = n. Since there are n terms, their sum is $n(n) = n^2$.

** http://nrich.maths.org/public/viewer.php?obj_id=325&part=solution&refpage=viewer.php