Solutions: Challenge Set 4

1. The greatest possible digit sum is 54 (999,999, there is only one integer with a digit sum of 54). Only one other digit sum is greater than 52 … there are six 6-digit integers whose digit sum is 53 (place an 8 in any one of the six places and fill the other five places with 9's). 1 + 6 = **7**.

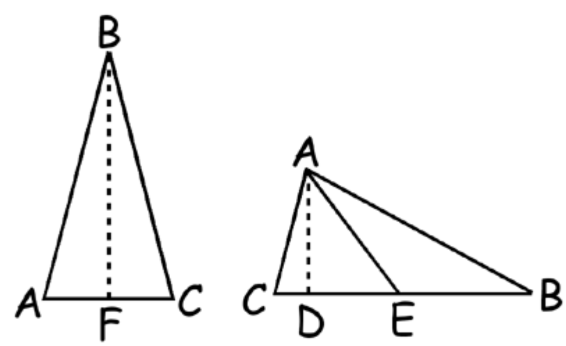
2. Fold your scratch paper and figure this one out on your own … 8 will be second from the bottom, 6 will be second from the top. 8 + 6 = **14**.

3. Given side lengths of the rectangle a and b, the area of the rectangle is ab and the diagonal length is $\sqrt{a^2+b^2}$, which gives us ab = 5 and $a^2 + b^2 = 15$. Now we use some clever manipulations:
   $(a + b)^2 =$
   $a^2 + 2ab + b^2 =$
   $(a^2 + b^2) + 2(ab) =$
   $15 + 2(5) = 25$.
   We see that $(a + b)^2 = 25$, so a + b = 5.
   This makes the perimeter 2(a + b) = **10**.

4. Aside from the standard method (guess-check), we can call the original tens digit a and the original units digit b. This makes 10a + b = 10b + a + 54, so we have 9a – 9b = 54, or a – b = 6. The difference in digits is 6, so we have 93, 82, 71, and 60. Of these, only **71** is prime.

5. Call the rate of travel of the plane r. With the wind, the plane travels (r + 20)mph and against the wind the plane travels (r – 20)mph. The distance there is the same as the distance back:
   Going there: d = 3(r + 20)
   Coming back: d = 4(r – 20)
   Therefore: 3(r + 20) = 4(r – 20) so 3r + 60 = 4r – 80 and r = 140.
   Plug this into either distance equation to get d = **480** miles.

6. First recognize that the congruent sides must be 4cm long (2-2-4 cannot be the sides of a triangle). Use the Pythagorean Theorem to find altitude BF = $\sqrt{15}$. Altitude AD is half the length of BF (because base CB is twice AC). Use AC = 2 and AD = $\sqrt{15}/2$ with the Pythagorean Theorem to find CD = $\frac{1}{2}$. DE is therefore 3/2 and we can use DE and AD to find AE = $\mathbf{\sqrt{6}}$. If there is a better way I haven't heard it, but please feel free to mail me an alternate solution.



7. We can call the number of crows 5x and the number of sparrows 12x for some number x. After one of each fly away the ratio is 2:5, so we can set-up the proportion: $\frac{5x-1}{12x-1} = \frac{2}{5}$. Solve for x to get x = 3. Initially there were 15 crows and 36 sparrows. After one of each flew away we had 14 and 35. If two more fly away we will have 12 and 33, for a ratio of **4/11**.

8. This one is actually much simpler than it initially appears. If eight cubes are placed together to form one larger cube, half of the faces of the original cubes will be on the outer surface and therefore

will receive paint (3 faces of each cube). The probability of initially getting red paint is $\frac{1}{2}$. The probability that a red-painted face will not be painted blue is also $\frac{1}{2}$. Each face is equally likely to end-up face-up when rolled, so we have $\frac{1}{2} \times \frac{1}{2} = \mathbf{\frac{1}{4}}$. Student Allan Jiang penned this the "journey of a single face", which I found to be quite an elegant term for this type of solution.

9. $720 = 2^4 \times 3^2 \times 5$ and therefore has $(4+1)(2+1)(1+1) = 30$ factors (see my book for a complete explanation of this factor trick). Each of those 30 factors is part of a pair whose product is 720 (for example, 72x10 or 20x36). There are $30/2 = 15$ pairs of factors, each with a product of $2^4 \times 3^2 \times 5$ so we have $(2^4 \times 3^2 \times 5)^{15} = 2^{60} \times 3^{30} \times 5^{15}$ which makes a + b + c = **105**.

10. Lots of isosceles right triangles make this one easy. The shaded triangle is an isosceles right triangle with hypotenuse 2 and area **1**u^2 ($\frac{1}{4}$ of a 2x2 square is one of the many ways to see this).