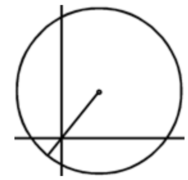
Solutions: Challenge Set 2

1. Plug-in x=3 and y=-3 to each equation and then solve for a and b: -3=3(3)+b gives us b = -12 and -3=3a+3 gives us a = -2. Add a + b to get **-14**.

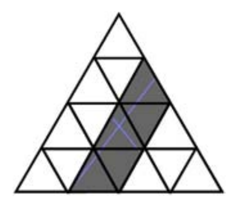


2. The center of the circle is 5 units from the origin (3-4-5 right triangle). If we draw a radius through the origin to its intersection with the circumference this gives us the point nearest the origin, which is 7-5=**2** units from the origin.

3. We will use a technique I call sticks and stones but is referred to elsewhere as balls and walls, stars and bars, balls and urns (indistinguishable balls, distinguishable urns), etc. Four of the pencils are already accounted for (1 to each student) so we are really dividing 6 pencils amongst 4 students. Using 0's to represent pencils, we can arrange six 0s with three dividers to split them between the students. For example, the diagram 0|00||000 represents giving one pencil to the first student, two to the second, none to the third, and three to the fourth student. This gives us 9 items to arrange (three dividers and 6 pencils). There are $_9C_3$ = **84** ways to place the three dividers in three of the nine positions. For a more thorough explanation, buy my book ☺.

4. In **12** minutes the bathtub will fill three times and drain twice (so it will be full after 12 minutes). It is also possible to calculate that the tub fills at a rate of 1/4 - 1/6 = 1/12 tub per minute, which leads us to the same answer.

5. We will calculate the probability that the same number will not appear on consecutive rolls. Whatever you roll first, there is a 5/6 probability of rolling something different on the second roll. Whatever you roll second, there is a 5/6 probability of rolling something different again. (5/6)(5/6) = 25/36 probability of not rolling the same number on consecutive rolls … making the probability of rolling the same number on consecutive rolls: 1 – 25/36 = **11/36**.



6. The diagram gives one way to see that the shaded portion is 5/16 the total area, or: The ratio of the small triangle (2" base) to the medium triangle (3" base) to the large triangle (4" base) is $2^2: 3^2: 4^2 = 4:9:16$. The shaded region is outside the small triangle but inside the medium triangle: 9-4=5 parts of the whole. The total area (large triangle) is 16 parts, so we again get **5/16**.

7. Powers of digits 1, 3, 7, and 9 have units digits that cycle (actually, all powers do). Any power of 1 ends in a 1. Powers of 3 cycle: 3-9-7-1-3-9-7-1… so every 4^{th} power of 3 ends in a 1. Powers of 7 cycle similarly 7-9-3-1-7-9-3-1… and powers of 9 cycle 9-1-9-1…, so each of these **4** digits raised to the 100^{th} power ends in a 1, making the remainder 1 when each is divided by 10.

8. Consider each 'lap'. Going N(1), W(2), S(3) and E(4) leaves us 2 miles south and 2 miles east of where we started the lap (as does each additional lap). To figure out how many 'laps' we take, we need to find n for 1+2+3+…+n = 210, so n(n+1)/2=210, giving us n=20. 20 'legs' means 20/4=5 laps, with each lap taking us 2 miles south and 2 miles east for a total of 10 miles south and 10 miles east. The straight-line distance is the hypotenuse of an isosceles right triangle with legs 10 miles long: $\mathbf{10\sqrt{2}}$ miles.

9. $\left(\frac{8}{27}\right)^{\left(-\frac{2}{3}\right)} = \left(\frac{27}{8}\right)^{\left(\frac{2}{3}\right)} = \left(\frac{\sqrt[3]{27}}{\sqrt[3]{8}}\right)^2 = \left(\frac{3}{2}\right)^2 = \mathbf{\frac{9}{4}}$.

10. Aside from a brute-force method, we recognize that numbers which are multiples of 2 and/or 3 make-up 2/3 of all integers. Notice the pattern: 1 ~~2~~ ~~3~~ 4 5 ~~6~~ 7 ~~8~~ ~~9~~ … (the four is crossed-out, but it is hard to tell with this font). This leads us to predict that if 60 numbers were written, then 20 numbers would be left un-erased. Of course, 60 would also be erased, so **59** is the 20th number that is not a multiple of 2 and/or 3 (and therefore not erased).**

**It was pointed out by two students that 23 could have been interpreted as the correct answer to the original problem as written below if the reading was that Nathan erased only numbers which were multiples of both 2 and 3 (multiples of 6). One student submitted an incorrect response of 24 due to a misreading and another submitted a 'backup' answer of 23 in case his reading is incorrect (23 would have been accepted with the original wording). I believe that the re-wording eliminates any ambiguity.

Original Problem:

10. ____________ Bridgett writes each of the first n positive integers on a board. After Nathan erases all of the multiples of 2 and 3, there are 20 numbers left. What is the smallest possible value of n?