# Fibonacci and Lucas Numbers

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# 1 Fibonacci Numbers

## 1.1 Sum of Terms

The sum of the first $n$ Fibonacci numbers is given as follows:

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

which we can easily prove with induction.

We use this to prove the following theorem:

**Theorem:** Let $n$ and $k$ be two positive integers. Prove that between the consecutive powers $n^k$ and $n^{k+1}$ there are no more than $n$ Fibonacci numbers.

**Proof:** We proceed by contradiction. Assume that between some $n^k$ and $n^{k+1}$, there exist at least $n+1$ Fibonacci numbers. Then we have,

$$n^k < F_{r+1}, F_{r+2}, F_{r+3}, \ldots, F_{r+n+1}, \cdots < n^{k+1}$$

The sum of the first $n-1$ of these numbers is

$$\begin{aligned} F_{r+1} + F_{r+2} + \cdots + F_{r+n-1} &= F_{r+n-1} + F_{r+n-2} + \cdots + F_1 - (F_r + F_{r-1} + \cdots + F_1) \\ &= F_{r+n+1} - 1 - (F_{r+2} - 1) \\ &= F_{r+n-1} - F_{r+2}. \end{aligned}$$

Solving for $F_{r+n+1}$ yields

$$F_{r+n+1} = (F_{r+1} + F_{r+2} + F_{r+3} + \cdots + F_{r+n-1}) + F_{r+2}$$

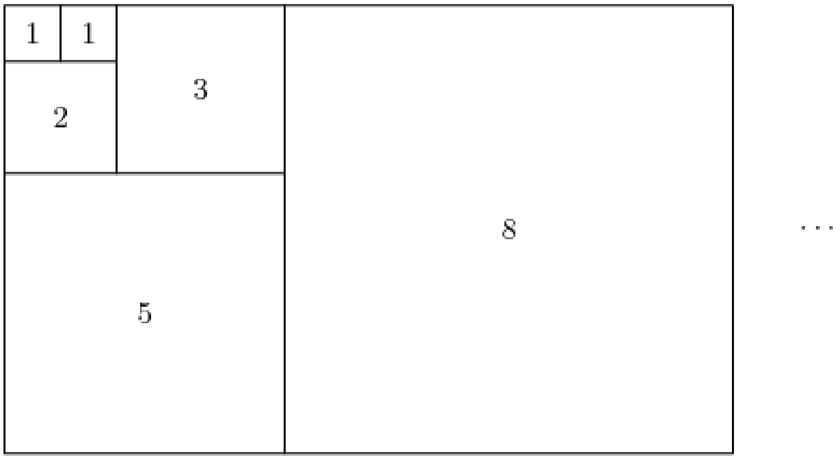
which is a sum of $n$ Fibonacci numbers, each of which is greater than $n^k$. Thus, $F_{r+n+1} > n(n^k) = n^{k+1}$, contradicting our assumption.

## 1.2 Sum of Squares

The formula for the sum of square of the first $n$ Fibonacci numbers is given as follows:

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$$

We can use induction to prove this, although there is a nice geometric proof for it. You should be able to figure it out with the diagram below:



$$\vdots$$

## 1.3 Binet's Formula

Binet's Formula states that $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$. We can elegantly derive this using the following lemma:

**Lemma:** If $x^2 = x + 1$, then, for $n = 2, 3, 4, \ldots$, we have:

$$x^n = F_n x + F_{n-1}$$

**Proof:** This is trivial for $n = 2$. Suppose that $x^n = F_n x + F_{n-1}$ for some $n > 2$. Then:

$$\begin{aligned} x^{n+1} &= x^n \cdot x = (F_n x + F_{n-1})x \\ &= F_n x^2 + F_{n-1} x \\ &= F_n(x+1) + F_{n-1} x \\ &= (F_n + F_{n-1})x + F_n \\ &= F_{n+1} x + F_n, \end{aligned}$$

as desired.

The two numbers $x$ which satisfy $x^2 = x + 1$ are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Thus, for all $n = 2, 3, 4, \ldots,$ we have

$$\alpha^n = F_n \alpha + F_{n-1}$$

and

$$\beta^n = F_n \beta + F_{n-1}.$$

Subtracting these yields $\alpha^n - \beta^n = F_n(\alpha - \beta)$. Noting that $\alpha - \beta = \sqrt{5}$ yields Binet's Formula.

# 2 Lucas Numbers

## 2.1 Introduction

Just as $F_n$ denotes the $n$th Fibonacci number, we define $L_n$ as the $n$th Lucas number. The Lucas sequence is defined by

$$L_n = F_{n-1} + F_{n+1}.$$

The first few Lucas numbers are $1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$.

Since the Fibonacci numbers are generated by the recursion

$$F_n = F_{n-1} + F_{n-2}$$

the Lucas numbers also have that property, that is, for $n > 2$,

$$L_n = L_{n-1} + L_{n-2}$$

We define $L_0 = 2$ because $L_2 = L_1 + L_0$.

Note that a Lucas number is always greater than its corresponding Fibonacci numbers, except for $L_1$.

## 2.2 A Formula

We use Binet's Formula to derive a formula for $L_n$. We have:

$$\begin{aligned} L_n &= F_{n-1} + F_{n+1} \\ &= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta}\left[\alpha^n\left(\frac{1}{\alpha} + \alpha\right) - \beta^n\left(\frac{1}{\beta} + \beta\right)\right]. \end{aligned}$$

Substituting $\alpha = \dfrac{1+\sqrt{5}}{2}$ yields $\dfrac{1}{\alpha} + \alpha = \sqrt{5} = \alpha - \beta$, and similarly, $\dfrac{1}{\beta} + \beta = -\alpha + \beta$, so the formula for the Lucas numbers is

$$L_n = \alpha^n + \beta^n.$$

## 2.3 Some Properties of Fibonacci and Lucas Numbers:

We can use Binet's Formula to help us prove the following theorem:

**Theorem:** Let $(1+2x)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Substitute $F_k$ for $x^k$ in this expression, yielding $a_0 F_0 + a_1 F_1 + a_2 F_2 + \cdots + a_n F_n$. This sum is equal to $F_{3n}$.

**Proof:** By the Binomial Theorem,

$$(1+2x)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k x^k.$$

Denote $S = a_0 F_0 + a_1 F_1 + a_2 F_2 + \cdots + a_n F_n$. Then,

$$\begin{aligned} S &= \sum_{k=0}^{n} \binom{n}{k} 2^k F_k \\ &= \sum_{k=0}^{n} \binom{n}{k} 2^k \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} \right) \\ &= \frac{1}{\alpha - \beta} \left[ \sum_{k=0}^{n} \binom{n}{k} 2^k \alpha^k - \sum_{k=0}^{n} \binom{n}{k} 2^k \beta^k \right] \\ &= \frac{1}{\alpha - \beta} [(1+2\alpha)^n - (1+2\beta)^n] \end{aligned}$$

Since $\alpha^2 = \alpha + 1$, we have

$$\begin{aligned} 1 + 2\alpha &= \alpha^2 + \alpha \\ &= \alpha(1+\alpha) \\ &= \alpha^3 \end{aligned}$$

Similarly, $1 + 2\beta = \beta^3$. Thus,

$$S = \frac{1}{\alpha - \beta}[(\alpha^3)^n - (\beta^3)^n] = \frac{\alpha^{3n} - \beta^{3n}}{\alpha - \beta},$$

which is just $F_{3n}$ by Binet's Formula.

We can continue with this. Because $L_n = \alpha^n + \beta^n$, we have:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k L_k = L_{3n}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} L_k = L_{2n}$$

Additionally, we have $F_{2n} = F_n L_n$, which we can prove using the formulas for $F_k$ and $L_k$.

Another theorem regarding the Lucas and Fibonacci numbers is:

**Theorem:**

$$F_{m+p} + (-1)^{p+1} F_{m-p} = F_p L_m.$$

This is easily proven.

**Proof:** We wish to show that

$$\frac{\alpha^{m+p} - \beta^{m+p}}{\alpha - \beta} + (-1)^{p+1} \frac{\alpha^{m-p} - \beta^{m-p}}{\alpha - \beta} = \frac{\alpha^p - \beta^p}{\alpha - \beta}(\alpha^m + \beta^m)$$

Then

$$\begin{aligned}\alpha^{m+p} - \beta^{m+p} + (-1)^{p+1}(\alpha^{m-p} - \beta^{m-p}) &= \alpha^{m+p} + \alpha^p\beta^m - \alpha^m\beta^p - \beta^{m+p} \\ (-1)^{p+1}(\alpha^{m-p} - \beta^{m-p}) &= \alpha^p\beta^m - \alpha^m\beta^p \\ (-1)(\alpha\beta)^p(\alpha^{m-p} - \beta^{m-p}) &= \alpha^p\beta^m - \alpha^m\beta^p \\ -(\alpha^m\beta^p - \alpha^p\beta^m) &= \alpha^p\beta^m - \alpha^m\beta^p \\ \alpha^p\beta^m - \alpha^m\beta^p &= \alpha^p\beta^m - \alpha^m\beta^p,\end{aligned}$$

as desired.

We use this to prove yet another property:

**Theorem:** The sum of any $4n$ consecutive FIbonacci numbers is divisible by $F_{2n}$.

**Proof:** Let $S$ equal this sum. Then we have:

$$\begin{aligned}S &= \sum_{k=a+1}^{a+4n} F_k \\ &= s_{a+4n} - s_a \\ &= (F_{a+4n+2} - 1) - (F_{a+2} - 1) \\ &= F_{a+4n+2} - F_{a+2}\end{aligned}$$

Letting $m = a + 2n + 2$ and $p = 2n$ yields

$$S = F_{a+4n+2} - F_{a+2} = F_{2n}L_{a+2n+2},$$

completing our proof.

One more theorem regarding the Lucas numbers:

**Theorem:** Suppose $p > 3$ is a prime number, and $p^k$ is a positive integral power of it. Then the $2p^k$th Lucas number $L_{2p^k}$ ends in a 3.

**Proof:** Numbers of the form $6m, 6m + 2, 6m + 3, 6m + 4$ are always composite for $m > 0$, so a prime $p > 3$ must satisfy

$$p \equiv \pm 1 \pmod 6$$

so

$$p^k \equiv \pm 1 \pmod 6$$

which implies $p^k = 6m \pm 1$ for some integer $m$. Thus, $2p^k = 12m \pm 2$.

The Lucas numbers end in the digits

$$1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, 4, 7, \ldots$$

which repeats with period 12:

$$1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2$$

The second and tenth numbers are 3's, so counting along to the $12m \pm 2$th Lucas number will always yield a units digit of 3.